Shear Deflections and Buckling Characteristics of Inflated Members

A. D. Topping*
Goodyear Aerospace Corporation, Akron, Ohio

The relation between shearing stiffness and inflation pressure is derived by taking into account beam edges and is used in buckling analyses of inflated plates and columns. The drop threads in inflated fabric beams redundantly affect the shear flow in the edges. From experiments on beams of coated metal cloth, it has been concluded from the small transverse curvature that in the flat portion of a beam the shear flow can be disregarded in many practical cases. Shear deflection equations are obtained both with this assumption and with drop threads disregarded. The results are applied to inflated columns by using the Euler criterion modified for shear flexibility. Similarly, conventional theory is extended to cover inflated plates. Eccentrically loaded pressurized-membrane columns are analyzed by modifying the secant formula for shear flexibility and conservatively assuming beam-column collapse to occur when local wrinkling begins.

Nomenclature

A = enclosed cross-sectional area
A' = cross-sectional area of solid beam
b = width of flat part of panel
c = distance from neutral axis to outermost fiber
E = Young's modulus of elasticity
e = initial eccentricity of axial load
G = shearing modulus of elasticity
h = depth of section at z
I = moment of inertia of cross section
K = buckling constant
k = section shape constant
L = column effective length
M = bending moment
N = skin tension
n = number of drop threads per unit area
P = axial load
p = inflation pressure
Q = first moment of cross section
q = shear flow
R = radius of edge or of cylinder
s = peripheral length of cross section
T_d = tension in a single drop thread
T = skin thickness
U = energy of shear deformation
V = transverse beam shear
w = distributed load per unit area
x, y, z = rectangular coordinates
α = slope of cross section in YZ plane
ϕ = angular slope of beam due to flexure
ν = Poisson's ratio
ρ = radius of gyration of cross section
θ = angle defining points at edges
ζ = angular slope of beam due to shear
λ = $pA + kGt$

Subscripts

cr = critical value
E = critical value neglecting shear flexibility
e = pertaining to edges
f = pertaining to flat faces
p = pertaining to pressure effect
U, L = pertaining to upper and lower surfaces, respectively
wr = at the first wrinkle

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* Senior Engineer Specialist (Stress). Member AIAA.

Introduction

In inflated plates and columns, shear flexibility is often significant. To understand the buckling characteristics of inflated members, therefore, their shearing behavior must be known.

Leonard, McComb, et al.[1–3] have developed a theory of inflatable structures, rigorously taking into account the coupling terms for combined bending and shear deflection but neglecting beam edges. Some other writers,[4–6] on the other hand, have completely neglected either shearing deflections or the influence of the inflation pressure upon them. For many pressure-stabilized structures this is satisfactory, but in some types of expandable structures, these latter assumptions can result in serious error. Foerster et al.[7] have shown that, in most practical structures, the coupling effect considered by Leonard, McComb, et al. may be neglected, so that a solution by simple superposition of bending and shearing deflections is possible. Effects of beam edges are readily included in this solution. This approach will therefore be taken here, since the edge shearing stiffness in some members is greater than the pressure stiffness.

Pressure Stiffness

The flexural stiffness of an inflated member is independent of the pressure, provided that the pressure is large enough to prevent wrinkling and that the yield strength of the material is not exceeded. The effect of pressure on the shearing deflection characteristics of thin-walled inflated prismatic beams, on the other hand, is often significant. It can be found as follows. Consider a strip of the beam, with a width dz, parallel to the longitudinal axis of the beam and to the plane of loading (Fig. 1). Let $Y - Y$ be a line in the plane of the element normal to the centerline of the strip at a point of zero shear. Let the angles ϕ and ζ describe the slopes of the beam due to flexure and to shear, respectively. Assume that the angles ϕ and ζ are small, so that $\sin\phi \approx \phi$, $\sin\zeta \approx \zeta$, and $\cos\phi \approx \cos\zeta \approx 1$. Summing vertical forces on the element, where w is the vertical component of a unit distributed load,

$$(dN_U + dN_L)(\phi + \zeta) - ph\,\phi\,dz = ds \int_0^x w\,dx \quad (1)$$

where dN_U and dN_L are the skin tensions on the widths $dz/\cos\alpha$.

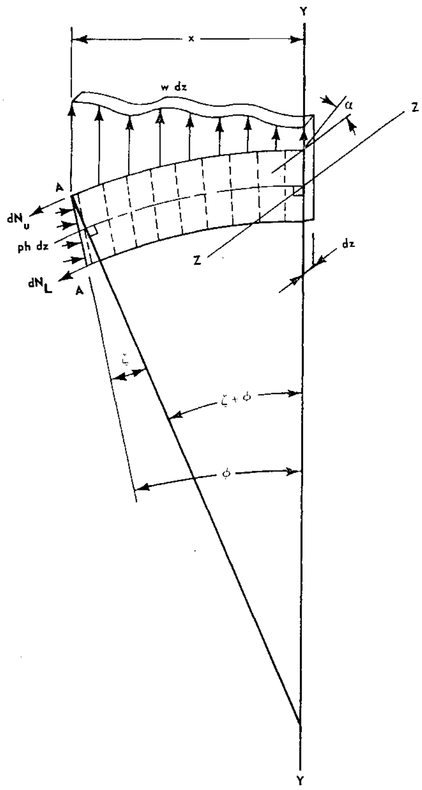

Fig. 1 Shear and bending geometry.

Equilibrium of the forces along the x axis at section $A - A$ requires that

$$dN_U + dN_L = ph\ dz \tag{2}$$

Substituting (2) into (1) yields

$$ph\ \zeta\ dz = dz \int_0^x w\ dz \tag{3}$$

If

$$\int_0^x w\ dx \int dz = V \qquad \int h\ dz = A \qquad \zeta = \frac{V}{pA} \tag{4}$$

Since the corresponding expression for shearing angular deflection of a solid beam, if the shearing force V is uniformly distributed over its cross section, is

$$\zeta = V/A'G \tag{5}$$

the pressure of the fluid inflating a beam can be treated as an effective shear modulus. This result was obtained from a different approach in Ref. 8 and agrees also with Refs. 1–3.

Effect of Edges

Practical inflated beams have shear resistant edges. The relative importance of the edges and the pressure can be examined in a simple example. Consider a flat inflated beam with circular edges and uniform skin thickness. This is a shape characteristic of AIRMAT,† in which closely spaced "drop threads" normal to the surface are used to maintain the shape. These drop threads have a significant effect upon shearing stiffness, as will be seen.

If we assume at first that the shape can be maintained without drop threads and that the flat part of the beam is fully effective, then the shear distribution is as shown in Fig. 2a, where V_e is that portion of the shear carried by the edges. For the round edges

$$q_e = \frac{V_e Q}{I} = \frac{2V_e\,(b + h\sin\theta)}{h(\pi h + 4b)} \tag{6}$$

and for the flat portion

$$q_f = q_{\theta=0}\left(\frac{2z}{b}\right) = \frac{4V_e\,z}{h\,(\pi h + 4b)} \tag{7}$$

The shear energy dU, absorbed in a length dx, is given by

$$dU = dx \int \frac{q^2\,ds}{2\,Gt} \tag{8}$$

Then, from Eqs. (6–8),

$$\frac{dU}{dx} = 4\int_0^{\pi h/4} \frac{2V_e^2(b + h\sin\theta)^2\,ds}{Gth^2(\pi h + 4b)^2} + 4\int_0^{b/2} \frac{8V_e^2\,z^2\,ds}{Gth^2(\pi h + 4b)^2} \tag{9}$$

In the first term, $ds = hd\theta/2$, and in the second, $ds = dz$. Integrating and putting

$$2k = \frac{3h^2\,(\pi h + 4b)^2}{6\pi b^2 h + 24bh^2 + 3\pi h^3 + 4b^3} \tag{10}$$

yields

$$dU/dx = V_e^2/2kGt \tag{11}$$

By using Castigliano's theorem, one obtains

$$\zeta_e = \frac{\partial}{\partial V_e}\left(\frac{dU}{dx}\right) = \frac{V_e}{kGt} \tag{12}$$

Considering now the shear V_p supported by the inflation pressure, one finds from Eq. (4), for this example,

$$\zeta_p = V_p/pA \tag{13}$$

The deflections ζ_e and ζ_p must be equal to the beam shear deflection ζ, and

$$V = V_e + V_p \tag{14}$$

Thus, from Eqs. (12–14), one finds for the shearing deflection

$$\zeta = \frac{V}{pA[1 + (kGt/pA)]} = \frac{V}{pA + kGt} \tag{15}$$

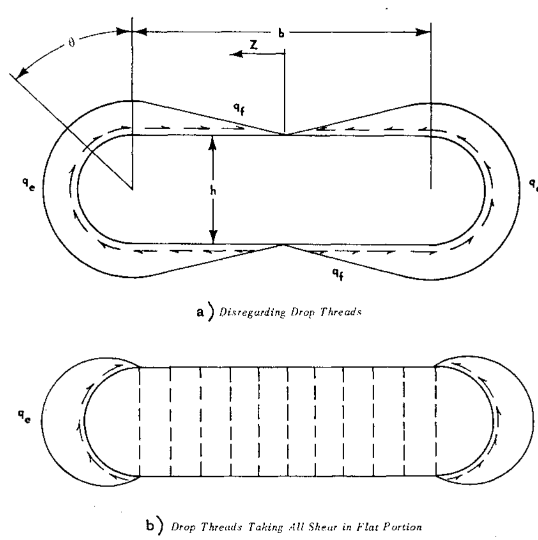

Fig. 2 Shear flow distribution in flat inflated beams.

† T. M. Goodyear Tire & Rubber Co., Akron, Ohio.

For a circular cylinder of radius R, $k = \pi R$, and (15) reduces to

$$\zeta = V/[\pi R\,(pR + Gt)] \tag{16}$$

Equation (16) shows that, in circular beams, the pressure effect is significant only when the hoop unit stress is of the same order of magnitude as the skin stiffness Gt. With isotropic materials this is unlikely, but it may occur with single-ply fabrics.[8]

Effect of Internal Drop Threads

For actual AIRMAT beams, the foregoing analysis is inadequate. Consider an element, as shown in Fig. 3, in which the drop thread tensions are treated as a uniformly distributed load normal to the skin before deformation. When pure transverse shear is applied to AIRMAT, the drop threads do not rotate. When pure bending is applied, they remain normal to the surfaces. In general, therefore, the angle between the drop threads and the normal to the skin is the shearing strain ζ. Summing forces along the x axis yields

$$dN_x dz - dq\,dx = nT_d \sin\zeta\,dx\,dz \tag{17}$$

Equations (6) and (7) tacitly assumed $T_d = 0$ and so evidently do not apply to AIRMAT.

If only small shear strains are considered, so that $nT_d \approx p$, (17) may be written

$$(dN_x/dx) - (dq/dz) = p\zeta \tag{18}$$

At some distance from the edges of a wide AIRMAT beam, dq/dz may reasonably be assumed to be small enough to neglect, so that the gradient of the longitudinal stresses is developed entirely by the drop thread tensions caused by the pressure. Then Eq. (18) can be reduced to Eq. (4) by putting $dN_x = dM/bh$ and $V = dM/dx$.

The general solution of Eq. (18) requires some consideration of the relative deflections of the edges and of the flat AIRMAT portion. In Ref. 7, it was observed that the transverse curvature of the beams tested was small, which implies that little shear was being transferred to the edges from the initially flat part of the beam. Accordingly, dq/dz was neglected over the entire flat portion of the beam, and the shear flow distribution of Fig. 2b was assumed.

Then Eqs. (6) and (7) become

$$q_e = 2V_e \sin\theta/\pi h \tag{19}$$

and

$$q_f = 0 \tag{20}$$

In view of (19) and (20), one may let $b = 0$ in Eq. (10), so that Eq. (12) becomes

$$\zeta_e = 2V_e/\pi Gth \tag{21}$$

Since ζ_p does not change, Eq. (15) becomes

$$\zeta = \frac{V}{pA\,[1 + (\pi hGt/2pA)]} \tag{22}$$

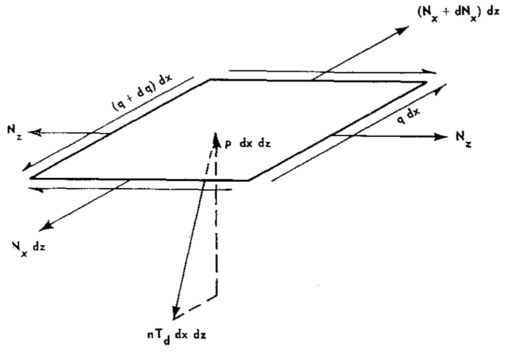

Fig. 3 Surface element of an AIRMAT beam.

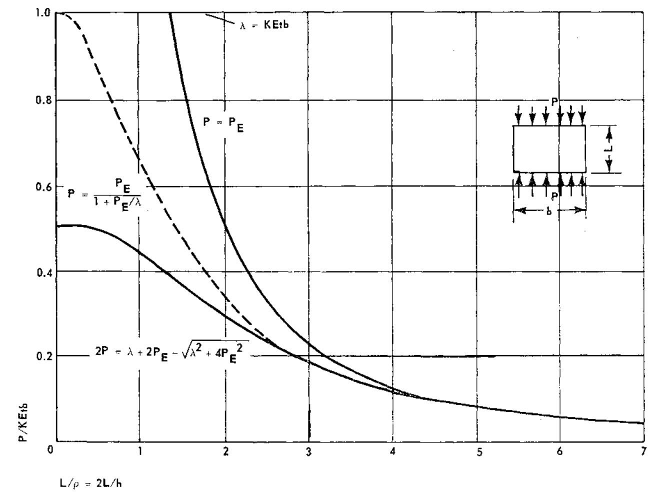

Fig. 4 Nondimensional compression buckling curves for inflated plates ($\lambda = pA + kGt = KEtb$).

which again reduces to (16) for a circular cylinder. Deflections calculated on the basis of Eq. (22) agree fairly well with measured deflections of AIRMAT beams in Ref. 7.

Column Stability

The stability of pressurized columns also depends on shear flexibility. It will be illuminating to limit the discussion to membrane structures in which the skin flexural stiffness is negligible. There are two basic criteria that an inflated membrane column must satisfy. The first is that the pressure must be great enough; i.e., if the longitudinal pressure stresses are uniformly distributed, then

$$pA \geqq P \tag{23}$$

The second is the Euler criterion, modified for shear flexibility with due regard for the effect of pressure (Timoshenko, Ref. 9, pp. 139–141):

$$P \leqq \frac{P_E}{1 + [P_E/(pA + kGt)]} \tag{24}$$

without drop threads, or with them if k is taken to be $\pi h/2$. Here

$$P_E = \pi^2 EI/L^2 \tag{25}$$

in which L is the distance between inflexion points of the column. Because the material of the column is all in tension, the necessity for Eq. (24) is a mild paradox. But the stress level in a Euler column has nothing to do with the critical buckling load except insofar as it influences deflections; the critical load is simply the axial load that will just hold the column in a bent position.

For a wide AIRMAT column, or plate loaded in compression on two opposite edges with the other two edges free, the radius of gyration

$$\rho \approx (I/2b)^{1/2} \approx h/2 \tag{26}$$

and it is necessary to introduce the factor $(1 - \nu^2)$, so that

$$P_E = \frac{\pi^2\,Etbh^2}{2L^2(1 - \nu^2)} \tag{27}$$

or

$$P_E/b = KEth^2/2L^2 \tag{28}$$

Equation (28) is plotted in Fig. 4 (upper curve).

The optimum design pressure, neglecting shear flexibility, would be obtained by equating the Euler load to pA, or

$$ph = P_E/b \tag{29}$$

From Eqs. (24) and (29), after disregarding the term kGt, it is clear that if shear flexibility is allowed for, no more than half the basic Euler load P_E can actually be supported with this pressure.

Effect of Axial Load

In the foregoing discussion, the effect of external axial loads on shearing stiffness, and the consequent effect of the change in shearing stiffness on buckling characteristics, was not taken into account. In compression members where P is not small compared to pA, this effect may be important; hence it is now considered.

If there is an external compressive axial load P, then in Eqs. (1–3), p must be replaced by the expression $(p - P/A)$, so that Eq. (4) becomes

$$\zeta = V/(pA - P) \tag{4a}$$

and Eq. (15) becomes

$$\zeta = \frac{V}{pA - P + kGt} \tag{15a}$$

Consequently, Eq. (24) should read as follows, since an axial load P certainly exists in a column:

$$P = \frac{P_E}{1 + [P_E/(pA - P + kGt)]} \tag{24a}$$

which, when solved for P after letting $pA + kGt = \lambda$, leads to the quadratic equation

$$P = \tfrac{1}{2}[\lambda + 2P_E \pm (\lambda^2 + 4P_E{}^2)^{1/2}] \tag{30}$$

Here only the negative sign of the radical is useful.

If a slenderness ratio is chosen such that $\lambda = KEtb$ (for example), and L is varied while the pressure p is held constant, the lower curve in Fig. 4 is obtained. Its equation is

$$P_{cr} = \frac{KEtb}{2}\left[1 + \frac{h^2}{L^2} - (1 + h^4/L^4)^{1/2}\right] \tag{31}$$

and it indicates the shape of the Euler curve, considering shear flexibility. (Its similarity to empirical column curves is interesting.) The limit $P = pA$ still exists at pA/λ, but since the maximum value for P/λ is 0.5 at $L/\rho = 0$, even this value may not be attainable. One can, in fact, conclude that an inflated membrane column can under no circumstances support an end compressive load equal to the internal pressure force on that end unless $kGt \geqq pA$.

A pair of curves similar to those in Fig. 4 can always be drawn for any problem, and if kGt is taken as zero, the lower curve always establishes a lower bound. The $P = P_E$ curve is an upper bound representing the case when kGt is large in comparison with P_E. Though the effect here may seem important only at very small values of L/ρ, with smaller values of λ, significant deviations from the elementary Euler theory will be found at much larger values of L/ρ.

Plate Theory

The buckling of plates without shear flexibility is described by equations of the following type (see, for instance, Timoshenko, Ref. 9 p. 329):

$$P_E/b = K'\pi^2 D/b^2 \tag{32}$$

where K' is a factor the magnitude of which depends on a/b, the aspect ratio of the panel, and

$$D = Eth^2/[2(1 - \nu^2)] \tag{33}$$

If K is the usual elastic buckling constant, (32) can be written

$$P_E/A = 6KEth/b^2 \tag{34}$$

Considering shear flexibility in inflated plates

$$P_{cr} = \frac{P_E}{1 + [P_E/(pA + kGt)]} \tag{35}$$

as in Eq. (24). If P/b is a compressive force per unit length, Eq. (24a) should be used instead, but (35) is applicable to shear loading as it stands. The term kGt may be neglected if the edges are supported. If Eq. (35) is plotted in Fig. 4 in a manner similar to Eq. (31) (assuming the same value of λ), it yields the dotted curve in Fig. 4.

Eccentrically Loaded Columns

If a column is loaded eccentrically or has an initial curvature, a somewhat different problem arises, since local instability in the presence of a bending moment is a more complex phenomenon than under axial load.[1,2,7] Under pure bending, the collapse moment for inflated members exceeds the moment necessary for first wrinkling by an amount that depends on the skin compressive stiffness and the shape of the cross section. For flat panels of fabric with little or no compressive stiffness, $M_{cr} \approx M_{wr}$, and this is a conservative assumption so long as the material strength is not exceeded.

In the presence of an axial load, then, and again assuming uniform stress distribution,

$$M_{cr} = \left(p - \frac{P}{A}\right)\frac{AI}{sc} \tag{36}$$

Let the moment arm of the eccentric load in the undeflected column be e. If there are no other loads, then from the secant formula (Ref. 9, pp. 14 and 5)

$$M = Pe \sec\left[\frac{\pi}{2}\left(\frac{P}{P_{cr}}\right)\right]^{1/2} \tag{37}$$

Because of shear flexibility, let P_{cr} be defined by Eq. (24). If M is the critical moment M_{cr}, Eqs. (36) and (37) give

$$(pA - P)\frac{I}{sc} = Pe \sec\left[\frac{\pi}{2}\left(\frac{P}{P_{cr}}\right)\right]^{1/2} \tag{38}$$

which can be solved by trial for the collapse load P.

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